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Existence of Equilibria and Complexity of Computation in Optimizing Complex Systems

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Abstract

This paper characterizes the generic properties of interior equilibria in complex systems. The field of complexity has been growing and is concerned with the complexity of computation of complex systems. Complex systems are characterized as systems with a large number of adaptive interdependent parts. These systems demonstrate several properties including: emergence, where the whole does not act as the sum of the parts, and sensitivity to initial conditions. As the number of economic factors increases, the recursion of input–output and modeling error propagates. These two particular symptoms of complex systems make them difficult to model. Solutions to optimization problems of this sort may have different properties depending on the functional assumptions. The generic properties of solutions to these problems can help us have some expectations over the complexity of computation. One example of a complex system is the Game Theoretic Interaction between economic agents. The paper also attempts to shed light on the complexity of computation of such systems.

Keywords: Complex Systems, Singularity Theory, Existence, Equilibrium, Optimization.

1. Introduction

The theory of complexity of computations is concerned with computational problems and evaluating their solution methods. Each method has its benefits and draws backs and is suitable to a specific kind of problem. This means that one can examine certain problems to determine which method performs best in terms of finding an accurate solution in feasible time for a type of problem. Problem types, however, need to be very specific for the method to perform for all instances of the problem. This poses a question: What are the generic properties of a problem and their solution sets that can be used to group problems into different levels of computational complexity[1]. For example, a problem that has a corner solution generically will not necessarily benefit from randomized search or steepest descent methods. This paper aims only to shed some light by a demonstrative example that uses singularity theory developed by Saari and Simone[2] to obtain generic properties of a solution space for optimization problems common in the static modeling of complex systems particularly the optimization of smooth differentiable functions.

Consider a system of 'n' parts all simultaneously optimizing. Question arises as to what are the generic properties of the solution to this particular problem and what are the consequences on the burden of computation when searching for it. This is often done by examples of particular problems, but they are often limited, because the properties of the solution space may change if the functional forms of the comprising system changes. It is useful to determine whether found properties of a solution concept hold good not just for a particular set of functions, but rather in general notion of being generic and "holding in general" usually means that the property holds for an open and dense set of utility functions, or at least for a countable intersection of open–dense sets.

The "denseness" condition ensures that it is general; because the closure of a dense set is everything, a property holding for a dense set is true almost everywhere. This allows us to make these generic statements about problems and their solutions.

Saari and Simon[2] developed a version of singularity theory to analyze the general properties of equilibria in the kinds of functions generally used in the social sciences viz, smooth differentiable functions. This basic approach is to find a general tractable method to describe static solutions of complex systems. The singularity theory approach relies on the properties of the implicit function theorem i.e., complex system equilibria defined by optima that are implicitly defined by the resulting first and second order conditions. Recovering the equilibrium strategies is essential, but only requires a standard tool, namely, the Inverse Function Theorem (IFT).

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To use the IMFT the optima must be defined in terms of the Space of Jets. The jet mapping takes a function 'F' and maps it into a space 'Jd' of degree 'd'. The space 'Jd' is a vector space that represents the domain, range, and up to 'F's dth derivative. In jet space, the higher order conditions imposed by the solution concept shape a surface in Euclidian Space. This surfaces' inverse image is the space of solutions. The inverse function theorem is powerful enough to tell us the dimension of the equilibrium space. This dimension can then be used to form an idea of how challenging the search would be for a solution.

Using this strategy yields an intuition that can be generalizable to classify the complexity of problems and the computation of their solutions. For example, the intuition that the number of constraints and unknowns will generically determine the dimension of the solution space is clear.

2. Jet Space

Consider the mapping F: $R^n \rightarrow R^m$, which has 'n' variables and 'm' equations f_i. Let J^1, the associated jet space with first order derivatives, be $J^1 = R^n \times R^m \times [R^n \times R^m]$. This can be considered as consisting of all possible domain points, all possible images, and all possible choices for the derivatives.

$\nabla F = (\nabla f_1, \nabla f_2, \ldots, \nabla f_m)$. A given map 'F' defines a mapping $j^1(F): R^n \subset J^1$ in the following natural manner: $j^1(F)(x) = (x, F(x), \nabla F(x))$. The equilibrium conditions we wish to impose on the function, when re–expressed in terms of J^1 variables, define the manifold $\Sigma \in J^1$.

To illustrate, suppose we are interested in the critical points of functions F: $R^2 \rightarrow R^1$. Here, $J^1 = ((x, y); z=F(x,y); (A_1, A_2))$, where $A_1 = \partial z/\partial x$, $A_2 = \partial z/\partial y \in R^1$. Critical point is where the partials of the function 'F' are zero. The equilibrium conditions can be re–expressed as the surface $\Sigma = ((x, y); z; (0, 0))$. If we could use the implicit function theorem, $j(f)^{-1}(\Sigma)$ would render all critical points of F.

Since, Game Theoretic Analysis is concerned with these special points, using jet space to describe optima may be a useful strategy.

Jet mapping can be used to transform the given information to space, that facilitates analysis key to the strategy. Just as important is to be able to take the answers back into the original space. This is done through the Inverse Function Theorem.

3. Inverse Function Theorem

The Inverse Function Theorem states that, if the total derivative of a function F: $R^n \rightarrow R^n$ is invertible at a point x (i.e., the determinant of the Jacobian of F at x is nonzero), and F is continuously differentiable near x, then it is an invertible function near x. That is, an inverse function to F exists in some neighborhood of F(x).

The Jacobian Matrix of F^{-1} at F (x) is then the inverse of the Jacobian of F, evaluated at p. We know that for a smooth mapping F: $R^n \rightarrow R^m$ and $x \in R^m$, then in general and locally $F^{-1}(x)$ is a n–m dimensional manifold.

So for n = m, expect $F^{-1}(x)$ to consist of isolated points. One can think of this as a system of equations and a number of unknowns. Since the number of equation is the same as the number of unknowns, one can easily find a unique solution. But, if m > n, in general, expect $F^{-1}(x)$ to be empty. Intuitively, this is clear because the number of equations is more than the number of unknowns. If n > m, expect $F^{-1}(x)$ to consist of n–m dimensional manifolds. It is clear that if the number of unknowns is bigger than the number of equations, then a system of equations has multiple solutions.

Instead of the inverse image of a point, suppose our interest is in the inverse image of a smooth manifold $\Sigma \subset R^m$ of dimension s. Here, the rank condition is replaced by a transversality condition. Namely, at a point $x \in \Sigma$, the linear space spanned by the tangent space $T_x\Sigma$ and the plane defined by:

$\nabla F(R^n)$ must have the full dimension 'm'. This shows as to why the results are local around the point of tangency. More importantly, the dimensionality of $F^{-1}(\Sigma)$ is n – [m – s]. Namely, the co–dimension of Σ defines the co–dimension of $F^{-1}(\Sigma)$. In other words, given that the transversality conditions are satisfied, the inverse function theorem will preserve the co–dimension of a smooth manifold.

4. Generic Transversality

To be precise for any given F, we would need to verify the transversality condition, but, if we are interested only in generic conclusions, then we are saved by an important result obtained by Thom[3]. By imposing an appropriate topology on the functions in function space, known as the Whitney Topology, Thom[3] proved for these jet mappings that, generically, either the mapping misses the target Σ or it meets it transversely. This means that once Σ is defined, if it can be established by some mapping that F allows its jet map to be in Σ, then generically the jet map meets transversely.

Definition 1: A smooth mapping F: $R^k \rightarrow R^m$ has a transverse intersection with a submanifold Σ of R^m if either (a) Image (F) T Σ = φ or (b) the condition of transversality: S pan[$D_x F(R^n)$ S $T_y\Sigma$] = R^m is satisfied for each X in F^{-1}.

We can now state Thoms[3] theorem.

Theorem 1 (Thom[3]): Let Σ be a smooth sub–manifold of R^m. Generically, a mapping F: $R^k \rightarrow R^m$ has a transverse intersection with Σ (i.e., this is true for a countable intersection of open dense sets in the space of such mappings F).

Consequently, all of the above co–dimension comments are established generically. Saari and Simon[2] developed singularity theory on the space of preferences to analyze the mathematical structure of Pareto sets. This paper aims to apply their methodology to complex optimizing systems.

5. System Equilibirium

General system equilibrium in a large system of optimizing interrelated parts is the solution of simultaneous maximization of each function with respect to all 'k' inputs. The optimal solution for part 'i', call it $X^* = \text{argmax}, F_j(X_j, X_{-i})$; $X_j = (x_{j1}, \ldots, x_{jk})$.

System Equilbrium occurs when all 'n' parts simultaneously optimize. Naturally, maximization imposes second order conditions for the negative definiteness of the Hessian Matrix.

To characterize this equilibrium using jet space, we use the $J_2 = R^{nk} \times R^k \times R^{nk} \times [R^{nk} \times R^{nk}]$, to write the system of equations in terms of the first derivative conditions and second derivative conditions.

$$J^2 = (X; Y; A_{11}, A_{12}, \ldots, A_{nk};\ B_{jil})\ \forall\ j, i, l$$
$$(X \in R^{nk}; Y \in R^n, A_{ji} \in R^{nk}, B_{jil} \in R^{nk} \times R^{nk})$$
$$j = 1,2,\ldots,n \quad i,l = 1,2,\ldots,k$$
$$X^T = (X_1, X_2, \ldots, X_n)$$
$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{pmatrix}$$

The matrix of inputs

$\nabla F(X) = (\nabla F_1(X), \nabla F_2(X) \ldots \nabla F_n(X))$

$$\nabla F_j(X) = \begin{pmatrix} \frac{\partial F_j(X)}{\partial x_{11}} & \frac{\partial F_j(X)}{\partial x_{12}} & \cdots & \frac{\partial F_j(X)}{\partial x_{1k}} \\ \frac{\partial F_j(X)}{\partial x_{21}} & \frac{\partial F_j(X)}{\partial x_{22}} & \cdots & \frac{\partial F_j(X)}{\partial x_{2k}} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial F_j(X)}{\partial x_{n1}} & \frac{\partial F_j(X)}{\partial x_{n2}} & \cdots & \frac{\partial F_j(X)}{\partial x_{nk}} \end{pmatrix}$$

$$\Sigma_2 = \{(X; Y; A_{ji}; B_{ji}) \mid A_{ji} = 0, B_{ji} < 0\ \forall i, j\} \qquad (1)$$

In this representation and with the jet map, $Y = [y_1, y_2, \ldots, y_n]$ and A_{ji} correspond to $\frac{\partial F_j}{\partial x_{ji}} \in \nabla F(X)$. The positive definiteness of Hessian Matrix insuring the optimization condition is represented through the correspondence between B_{ji} and $\frac{\partial^2 F_j}{\partial x^2_{ji}} \in \nabla^2 F(X)$.

Then the co–dimension of Σ_2 in the jet space will be 'r' which is the number of closed restrictions in the space of Jets. Consider the co–dimension of optimality restrictions implied by the first order condition $A_1, \ldots, A_n$. There are 'nk' restrictions. Given Generic Transversality, the Inverse Function Theorem preserves the co-dimension. By the findings of Saari[4], to back out the dimension of the equilibrium subspace, we compare the dimension of the domain 'nk' to the co–dimension 'r' of Σ_2 in the jet space J_2.

Since, $r = n^*k$, the co–dimension of the critical points in strategy space is $nk - nk = 0$. This means that generically, the interior equilibria of a complex optimizing system are isolated points so they do not trace a curve.

Theorem 2: For a complex system S(N,F,X), where N is the set of optimizing system parts and contains n>1 parts, F is differentiable and $\nabla F(X) = (\nabla F_1(X), \nabla F_2(X), \ldots, \nabla F_n(X))$ and $X^T = (X_1, X_2, \ldots, X_n))$ where $X_j = [x_{11}, x_{12}, \ldots, x_{1k}]$, generically, interior system equilibria are isolated points and the space of boundary solutions is of dimension nk.

Proof 1: d = Dimension of the Input Space Co–dimension (Σ_2).
$d = nk - n^*k = 0$.

Notice that at the boundary points the first and second order conditions are not binding. Then the equilibrium subspace, Σ, has to be rewritten as: $\Sigma_0 = (X; F(X))$ Therefore, $d_{boundary} = nk$.

In the boundary solution case, there is not a single constraint. Consequently, in general, the conditions for boundary point solutions can be expected to be (locally) satisfied along some collection of curves in R^{nk}. However, this proves that generically, interior system equilibria exists and are isolated points. In other words, if X^* is a critical point for F, then in any sufficiently small neighborhood of X^* there are no other critical points of F (X). This also means that imposing any other restrictions may affect the existence of the equilibrium.

The next section considers the consequences of this result on computation and searching for a solution.

6. Complexity

In this section the paper will examine both the complexity of systems and complexity of computation. Generally speaking, as more parts are introduced to the system, and as the number of inputs increase, the system becomes more complex. However, when concerned with the searching of the solution of the system optimization, we can use the results of this paper to give more guidance to computational complexity. Computational complexity of optimizing complex system would depend on the search space and on the dimension of the solution space.

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If the solution space covers the search space then the problem is quite simple. If the solution space is a two dimensional curve in a large three dimensional search space, then it is the proverbial needle in a hay stack. If as in the theorem above, the solution space is a point in the search space, then generically it is a grain in a haystack. For example, generically in large spaces the search time may be long to locate these isolated points longer than if the solution space traced a curve for example. More importantly, any additional binding restrictions would generically jeopardize the existence of the solution meaning the search algorithm would be wasting its time searching for an interior equilibrium when the right search would be on the boundary.

7. Conclusions

Since much of the research on complexity of computation is being done on examples of problems, especially when trying to look at **P** versus **NP**, the analysis suggests that generic properties of solutions (for some problems) the number of binding constraints may make the solutions space so small compared the search space it may lead to large search time. This suggests that computational complexity could be the difference between the dimension of the search space and dimension of the equilibrium subspace Σ. When d = nk, the complexity of computation C(d) = 0. When d is negative, the interior solution does not exit; then a interior search would take up valuable computation time and only ends with the probability depending on the number of searched points in the area of the boundary of nk−1 dimensions over the total number of searchable points in the space of nk dimensions. Likewise, when d > 0, the number of points in the solution subspace $\in R^{nk}$ over the number of points in the search space of dimension R^{nk} would be relevant to finding a solution.

8. References

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