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Survey Article on Comultiplication Modules

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Abstract

In this paper we will discuss some concrete results of comultiplication modules.

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1. Introduction

The concept of comultiplication modules was first introduced by H. Ansari-Toroghy and Farshadifar[4] in 2007. They used the word "comultiplication module" to those R-modules M for whom every submodule is the annhilator of some ideal of the ring R. They defined the comultiplication module by the following way:

An R-module M is called comultiplication module if for every submodule N of M , there exists an ideal I of R such that

$N = ann_M (I)$.

They also proved that an R-module M is comultiplication module if and only if for any submodule N of M , $N = (0 :_M ann_R (N))$. They proved that every proper submodule of a comultiplication module is comultiplication module. However, converse may not be true. For example, if V is a two dimensional vector space over a field k then V cannot be comultiplication module but every proper subspace of V is comultiplication as every one dimensional vector space is comultiplication module.

In 2008, Ansari and Farshadifar[7], further extended their work done in[4]. In this paper they gave a characterization of comultiplication modules in terms of completely irreducible submodules. They also proved that if R is local ring then every comultiplication module is cocyclic. Further they proved that a finitely generated second submodule of a comultiplication module is multiplication module. It was also shown that every non zero comultiplication module contains a minimal submodule and a characterization of minimal submodules was also given.

In the same year, Ansari and Farshadifar[6] further extended their work and proved that, over a Noetherian ring, an injective multiplication modules is comultiplication. They also proved the dual notion of Nakayama's lemma for finitely cogenerated modules.

In 2009, Atani and Atani[2] studied the comultilication modules over Dedekind domains and pullback of local Dedekind domains. They characterized the comultiplication modules over Dedekind domains with the help of localization. They completely described the indecomposable comultiplication modules over pullback of local Dedekind domains. This description was given in two stages. In the first stage they described the separated indecomposable comultiplication module and proved that if M is any separated indecomposable comultiplication module over a pull- back ring R of local Dedekind domains R_1 & R_2 , then M is isomorphic to one of the following modules:

(1) $M = (E(R_1/m1) \rightarrow (0) \leftarrow (0)), ((0) \rightarrow (0) \leftarrow E(R_2 /m_2))$, where $E(R_i/m_i)$ is the R_i-injective hull of R_i/m_i for all $i = 1, 2$.

(2) $M = (R_1 /m_1^n \rightarrow \bar{R} \leftarrow R_2/m_2^m)$.

In second stage they explained the non-separated indecomposable comultiplication module.

In 2011, Yousef Al-Shaniafi and Patrick F. Smith[12] studied the localization of comultiplication modules over a general ring R and proved that if every maximal ideal m of R is good for M then M is comultiplication R-module if and only if M_m is a comultiplication R_m-module for every maximal ideal m of R. It was also shown that if an R-module $M = \bigoplus_{i\in I} U_i$ is the direct sum of simple submodules $\{U_i\}i\in I$ for some index set I , then M is comultiplication module if and only if $\cap_{j\neq i} ann_R (U_j) \not\subset ann_R (U_i)$ for all $i\in I$. Further, they had shown that, under certain circumstances, quasi-injective modules and comultiplication module are related. They proved that if R is any ring and if M is a Noetherian quasi-injective R-module, then M is comultiplication module if and only if $Rx = (0 :_M ann_R (Rx))$ for all $x\in M$.

In the same year Ansari and Fashadifar[8,5] continued their work done in[4,7,6]. In[8], they proved that if every proper submodule N of a module M is comultiplication module and if $Ann_R (N) \neq Ann_R (M)$, then M is a comultiplication module. In[5], they proved that every second submodule of a Noetherian comultiplication module is simple submodule.

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In 2012, Al-Shaniafi and Smith[9] explained the minimal completely irreducible submodules, unique complements and Goldie dimensions of comultiplication modules. In[9], they proved that every comultiplication module has unique complement and if R is semilocal ring with n distinct maximal ideals, then every comultiplication mod- ule has Goldie dimension at most n. They also extended some results of Quasi-injective module and proved that every Noetherian comultiplication R-module is an Artinian quasi- injective R-module. In the same year, Tuganbaev[1] studied the comultiplication modules over non-commutative rings.

2. Characterization of Comultiplication Modules

Proposition 2.1. [[12], Proposition 1.3] Let R be a ring and let M be an R-module. Then M is comultiplication module if and only if for every submodule N of M such that M/N is cocyclic, there exists an ideal I of R such that $N = \text{ann}_M(I)$.

Proof. If M is comultiplication R-module then for every submodule N of M there exists an ideal I of R such that $N = \text{ann}_M(I)$ and hence the result follows.

Conversely, suppose that there exists an ideal I of R such that $N = \text{ann}_M(I)$ for any submodule N of M with M/N is cocyclic.

Let L be any proper submodule of M . Then by [[10], pp 2], there exists $\{L_j\}_{j\in J}$ of completely irreducible submodules of M such that $L = \cap_{j\in J} L_j$ and the module M/L_i is cocyclic for all $i \in \Delta$.

Now, by assumption, for every $i \in \Delta$, there exists an ideal J_i of R such that $L_i = \text{ann}_M(J_i)$.

Therefore,

$$L = \cap_{i\in\Delta} L_i = \cap_{i\in\Delta} \text{ann}_M(J_i) = \text{ann}_M\left(\sum_{i\in\Delta} J_i\right) = \text{ann}_M(K)$$

where $K = \sum_{i\in\Delta} J_i$, is an ideal of R.

Hence M is a comultiplication module.

Theorem 2.2. [[12], Theorem 1.5] For any R-module M , the following are equivalent.

(1) M is a comultiplication module.
(2) $N = (0 :_M \text{ann}_R(N))$ for every submodule N of M .
(3) The module $(0 :_M \text{ann}_R(_N))/N$ has zero socle for every submodule N of M .
(4) Given submodule P , L of M , $\text{ann}_R(P) \subseteq \text{ann}_R(L)$ implies that $L \subseteq P$.
(5) Given any submodule N of M and $x \in M$, $\text{ann}_R(N) \subseteq \text{ann}_R(Rx)$ implies that $x \in N$.
(6) Given any submodule N of M and $x \in M$, $\text{ann}_R(N) \subseteq \text{ann}_R(Rx)$ implies that $(N :_R x)$ is not a maximal ideal of R.
(7) $(L :_R N) = (\text{ann}_R(N) :R\, \text{ann}_R(L))$ for all submodules L and N of M .
(8) M is strongly self-cogenerated.

Proof. (1)⇔(2)

Let M be a comultiplication module, then for any submodule N of M , there exists an ideal I of R such that $N = \text{ann}_M(I)$. This implies that I .N = 0. Therefore, $I \subseteq \text{ann}_R(N)$, implies that $(0 :_M \text{ann}_R(N)) \subseteq \text{ann}_M(I) = N$. Obviously we always have $N \subseteq (0 :_M \text{ann}_R(N))$. Therefore,

$$N = (0 :_M \text{ann}_R(N)).$$

Conversely, let M is an R-module and N is a submodule of M. Now, $\text{ann}_R(N)$ is an ideal of R and $N = (0 :_M \text{ann}_R(N))$. Hence by definition, M is a comultiplication module.

(2)⇔(3)

(2)⇒(3) is quit obvious. We only need to prove the converse. So, suppose that for every submodule N of M , the module $(0 :_M \text{ann}_R(N))/N$ has zero socle. If possible, suppose that $N = (0 :_M \text{ann}_R(N))$ for some submodule N of M . Then by [[12], Lemma 1.4], there exists a submodule P containing N such that $(0 :_M \text{ann}_R(P))/P$ has non zero socle. But this contradicts our initial assumption. Therefore, $N = (0 :_M \text{ann}_R(N))$.

(2)⇒(4)

Let $N = (0 :_M \text{ann}_R(N))$ for all submodule N of M . Let P and L be submodule of M such that $\text{ann}_R(P) \subseteq \text{ann}_R(L)$. Let $x \in L$. This implies that $x \in (0 :_M \text{ann}_R(L))$ implies that $\text{ann}_R(L).x = 0$. Hence $\text{ann}_R(L) \subseteq \text{ann}_R(Rx)$, that is, $\text{ann}_R(P) \subseteq \text{ann}_R(L) \subseteq \text{ann}_R(Rx)$. This implies that $Rx = (0 :_M \text{ann}_R(Rx)) \subseteq (0 :_M \text{ann}_R(P))$, that is, $x \in P$. Therefore, $L \subseteq P$.

(4)⇒ (5) is obvious. (5) ⇒ (6)

Suppose for any submodule N of M , and $x \in M$ such that if $\text{ann}_R(N) \subseteq \text{ann}_R(x)$, then $x \in N$. Therefore, $(N :_R x) = R$.

(6)⇒ (2)

Suppose (6) holds. Let N be any submodule of M such that for any $x \in M$, $\text{ann}_R(N) \subseteq \text{ann}_R(Rx)$. By hypothesis, $(N :_R x)$ is not a maximal ideal of R. If possible, suppose, there is a submodule L such that

$$L \neq (0 :_M \text{ann}_R(L)).$$

Note that $L \subset (0 :_M (0 :_R L))$. Since $L = (0 :_M (0 :_R L))$, choose $x \in (0 :_M (0 :_R L))$ such that $x \notin L$. This implies that $\text{ann}_R(L) \subseteq \text{ann}_R(Rx)$.

Let $\mathfrak{F}$ be the family of all submodules x of M containing N such that $\text{ann}_R(X) \subseteq \text{ann}_R(Rx)$ and $x \notin X$. Then $\mathfrak{F}$ is non-empty implies that $L \in \mathfrak{F}$. Now suppose $\{X_i | i\in\Delta$, where Δ is an index set$\}$ be any chain in $\mathfrak{F}$. Put $X = \cup_{i\in\Delta} X_i$. Then x is a submodule of M containing L such that

$$\text{ann}_R(X) \subseteq \text{ann}_R(X_i) \subseteq \text{ann}_R(Rx) \text{ for all } i \in \Delta.$$

Thus $x \in \mathfrak{F}$ and is an upper bound of $\{X_i | i \in \Delta\}$. Therefore by Zorn's lemma $\mathfrak{F}$ admits a maximal element. Let P be any maximal element of $\mathfrak{F}$. As $L \subseteq P$, we have $\text{ann}_R(P) \subseteq \text{ann}_R(L)$ and hence

$$(0 :_M \text{ann}_R(L)) \subseteq (0 :_M \text{ann}_R(P)).$$

Therefore, $x \in (0 :_M \operatorname{ann}_R (P))$ and $x \notin P$. Since $(P :_R Rx)$ is a proper ideal of R, choose $a \in R$ such that $a \notin (P :_R Rx)$, that is, $ax \notin P$. Therefore, we conclude that $P + Rax \notin \mathfrak{F}$. Also as

$\operatorname{ann}_R (P + Rax) \subseteq \operatorname{ann}_R (P) \subseteq \operatorname{ann}_R (Rx)$,

we have $x \in P + Rax$, that is, $x = y + bax$ for some $y \in P$, $b \in R$, implies that $(1-ba)x = y$. This implies that $1 - ba \in (P :_R Rx)$. Therefore, $(P :_R Rx)$ is a maximal ideal of R. But this contradicts our initial hypothesis. Hence $N = (0 :_M (0 :_R N))$ for every submodule N of M.

(2)⇒(7)

Suppose (2) holds. Let L be any submodule of M and let $I = \operatorname{ann}_R (L)$. Note that $r \in (L :_R N)$ if and only if $rN \subseteq L = (0 :_M \operatorname{ann}_R (L)) = \operatorname{ann}_M (I)$, that is, $rIN = 0$ if and only if $rI \subseteq \operatorname{ann}_R (N)$, that is, $r \in ((0 :_R N) :_R \operatorname{ann}_R (L))$. Therefore,

$(L :_R N) = (\operatorname{ann}_R (N) :_R \operatorname{ann}_R (L))$.

(7)⇒(4)

Suppose (7) holds. Let P and L be submodules of M such that $\operatorname{ann}_R (P) \subseteq \operatorname{ann}_R (L)$. By hypothesis, $(P :_R L) = (\operatorname{ann}_R (L) :_R \operatorname{ann}_R (P)) = R$. Therefore, $L \subseteq P$.

Since equivalence of (4) and (2) is already established, we have, (7)⇒(2).

(1)⇒(8)

Suppose M is a comultiplication module and N be any submodule of M. Then, there exists an ideal I of R such that $N = \operatorname{ann}_M (I)$. Now, for every $a \in I$, define a trivial endomorphism $\phi_a : M \to M$ by $\phi_a (x) = ax$ for all $x \in M$. Obviously, we have $N = \cap_{a\in I} \ker \phi_a$. Therefore, M is strongly self-cogenerated module.

(8)⇒ (1)

Suppose (8) holds. Let L be any submodule of M. By hypothesis, there exists an index set J and trivial endomorphisms $\{\theta_j\}_{j\in J}$ on M such that $L = \cap_{j\in J} \ker \theta_j$. Since every endomorphism θ_j is trivial. Hence for every $j \in J$, there exists $a_j \in R$ such that

$\theta_j (x) = a_j x$ for all $x \in M$.

Suppose that $I = \sum_{j\in J} Ra_j$. Then

$x \in \operatorname{ann}_M (I) \Leftrightarrow x \in \cap_{j\in J} \operatorname{ann}_M (Ra_j) = \cap_{j\in J} \ker \theta_j = L$.

Therefore, $L = \operatorname{ann}_M (I)$ and hence, M is a comultiplication module.

3. Properties of Comultiplication Modules

Proposition 3.1. [7, Proposition 3.1] The following results hold for a comultiplication R-module M.

(1) If J is an ideal of R such that $\operatorname{ann}_M (J) = (0)$, then $JM = M$.
(2) If J is an ideal of R such that $\operatorname{ann}_M (J) = (0)$, then for every element $x \in M$, there exists an element a of J such that $x = ax$. In particular $Rx = Jx$ for all $x \in M$.
(3) If M is a finitely generated R-module and J is an ideal of R such that $\operatorname{ann}_M (J) = (0)$, then there exists $a \in J$ such that $1 - a \in \operatorname{ann}_R (M)$.

Proof. (1) Let N be any submodule of M. Then there exists an ideal I of R such that $N = \operatorname{ann}_M (I)$.

Let $x \in (\operatorname{ann}_M (I) :_M J)$.

$$\begin{aligned} &\Leftrightarrow Jx \subseteq \operatorname{ann}_M (I)\\ &\Leftrightarrow IJx = (0)\\ &\Leftrightarrow Ix \subseteq \operatorname{ann}_M (J) = (0)\\ &\Leftrightarrow x \in \operatorname{ann}_M (I). \end{aligned}$$

Therefore, $\operatorname{ann}_M (I) = (\operatorname{ann}_M (I) :_M J)$ and hence $N = (N :_M J)$. Put $N = JM$. Therefore,

$JM = (JM :_M J) = M$.

(2) Suppose that $x \in M$. Then Rx is a submodule of M. As $\operatorname{ann}_M (J) = (0)$, we have $\operatorname{ann}_{Rx} (J) = (0)$. By [12, Lemma 2.1], Rx is comultiplication R-module. Therefore, $Rx = Jx$, by (1) and hence result follows. This implies that $1x = ax$ for some $a \in J$.

(3) Let J be an ideal of R such that $\operatorname{ann}_M (J) = (0)$. Then by (1), $JM = M$. Now, since M is finitely generated R module and $JM = M$, hence by Nakayama Lemma, we have $1 - a \in \operatorname{ann}_R (M)$ for some $a \in J$.

Theorem 3.2. [7, Theorem 3.4] Let M be a faithful comultiplication R-module.

(1) If M is finitely generated module then $\operatorname{ann}_M (I) = (0)$, for every proper ideal I of R.
(2) If $\operatorname{ann}_M (m) \neq (0)$, for every maximal ideal of R then M is finitely cogenerated.

Proof. (1) Let N be finitely generated submodule of M. If possible, suppose that I is a proper ideal of R such that $\operatorname{ann}_M (I) = (0)$. Since I is a proper ideal, $I \subseteq m$, for some maximal ideal m of R. This implies that $\operatorname{ann}_M (m) \subseteq \operatorname{ann}_M (I) = (0)$. Hence by Proposition 3.1(3), $1 - a \in \operatorname{ann}_R (M)$, for some $a \in m$. Since M is faithful, we have $\operatorname{ann}_R (M) = (0)$. This implies that $a = 1 \in m$, which is a contradiction. Hence $\operatorname{ann}_M (I) = (0)$ for any proper ideal I of R.

(3) Let $\operatorname{ann}_M (m) = (0)$ for every maximal ideal m of R. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a collection of submodules of M such that $\cap_{\lambda\in\Lambda} M_\lambda = (0)$. Since M_λ is a submodule of M, for every $\lambda \in \Lambda$, there exists an ideal I_λ of R such that $M_\lambda = \operatorname{ann}_M (I_\lambda)$. Now,

$(0) = \cap_{\lambda\in\Lambda} M_\lambda = \cap_{\lambda\in\Lambda} \operatorname{ann}_M (I_\lambda) = \operatorname{ann}_M (\sum_{\lambda\in\Lambda} I_\lambda)$.

Note that $\sum_{\lambda\in\Lambda} I_\lambda$ is an ideal of R. We assert that $\sum_{\lambda\in\Lambda} I_\lambda = R$. If possible, suppose that $\sum_{\lambda\in\Lambda} I_\lambda \neq R$. Then $\sum_{\lambda\in\Lambda} I_\lambda \subseteq m$ for some maximal ideal m of R. But this implies that $\operatorname{ann}_M (m) \subseteq \operatorname{ann}_M (P I_\lambda) = (0)$, which is a contradiction. Therefore, $\sum_{\lambda\in\Lambda} I_\lambda = R$. Since $1 \in R$, there exists a finite subset Λ_1 of Λ such that $1 = \sum_{\lambda\in\Lambda_1} r_\lambda$, where $r_\lambda \in I_\lambda$. Therefore, $R = \sum_{\lambda\in\Lambda_1} I_\lambda$.

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Now, $ann_M(R) = (0)$. This implies that $ann_M(\sum_{\lambda\in\Lambda_1} I_\lambda) = (0)$, that is, $\cap_{\lambda\in\Lambda 1} ann_M(I_\lambda) = (0)$, that is, $\cap_{\lambda\in\Lambda 1} M_\lambda = (0)$. Hence M is finitely cogenerated.

Example 3.3. [7, Example 3.8] Let n be a fixed positive integer. Then

(1) Z_n is a comultiplication Z-module.
(2) Z_n is a comultiplication Z_n-module.

Proof. We prove only (1). The proof of (2) is same as that of (1).

Let N be a submodule of Z_n. Let $o(N) = d$. Then $n = md$ for some positive integer m. This implies that $N = mZ_n$. Put $I = dZ$. Then dZ is an ideal in Z such that

$$N = ann_{Z_n}(dZ).$$

Therefore, Z_n is a comultiplication Z-module.

4. Quasi-injective Comultiplication Modules

Theorem 4.1. [12, Theorem 4.4] Let R be any ring and let M be a Noetherian R-module such that

(1) $Rx = (0 :_M ann_R(Rx))$ for all $x \in M$ and
(2) $ann_R(N \cap P) = ann_R(N) + ann_R(P)$ for all submodules N and P of M .

Then M is quasi injective.

Proof. Let M be Noetherian R-module such that (1) and (2) holds. Since M is Noetherian module, every submodule of M is finitely generated. Now,

$$Rx = (0 :_M ann_R(Rx)) \text{ for all } x \in M$$

and let N and P are finitely generated submodules of M such that

$$ann_R(N \cap P) = ann_R(N) + ann_R(P).$$

Let $\beta : L \to M$ be an R-homomorphism. Then by [12, Lemma 4.3], there exists $r \in R$ such that

$$\beta(x) = rx \text{ for all } x \in L.$$

Therefore, β can be lifted to M , naturally by defining $\beta(x) = rx$ for all $x \in M$. Hence M is quasi-injective module.

Proposition 4.2. [9, Corollary 3.12] Every Noetherian comultiplication R-module is an Artinian quasi-injective R-module.

Proof. Let L be any submodule of M. Then by [11, Proposition 6.2], L is finitely gener- ated. Also, as M is comultiplication module, by [9, Corollary 3.11], every homomorphism $\phi : L \to M$ is trivial. Hence, $\phi : L \to M$ can be lifted to M . Therefore, M is M-injective and hence quasi-injective. Now, by [12, Corollary 2.11], M is Artinian. Therefore, M is Artinian quasi-injective module.

5. Comultiplication Module over Dedekind Domain

Lemma 5.1. [2, Lemma 3.2] Let R be the pullback ring. Then the indecomposable separated comultiplication module over R are

(1) $M = (E(R_1/m_1) \to (0) \leftarrow (0)), ((0) \to (0) \leftarrow E(R_2/m_2))$, where $E(R_i/m_i)$ is the R_i-injective hull of R_i/m_i for all i = 1, 2.
(2) $M = (R_1/m_1^n \to \bar{R} \leftarrow R_2/m_2^m)$.

Proof. Let R be the pullback ring and let $M = (M_1 \to \bar{M} \leftarrow M_2)$ be separated R-module. Then by [3, Lemma 2.8],

$M = (E(R_i/m_i) \to (0) \leftarrow (0)), ((0) \to (0) \leftarrow E(R_2/m_2))$ and
$M = (R_1/m_1^n \to \bar{R} \leftarrow R_2/m_2^m)$

are indecomposable. Now, by [2, Theorem 2.5],

R_1/m_1^n, R_2/m_2^m and $E(R_1/m_1)$, $E(R_2/m_2)$

are comultiplication modules. This implies by [2, Propo-sition 3.1],

$M = (E(R_1/m_1) \to (0) \leftarrow (0)), ((0) \to (0) \leftarrow E(R_2/m_2))$ and
$M = (R_1/m_1^n \to \bar{R} \leftarrow R_2/m_2^m)$

are comultiplication R-modules.

Lemma 5.2. [2, Lemma 2.4] Every non-zero comultiplication module over a discrete valuation domain R is indecomposable.

Proof. Let R be a discrete valuation domain with $m = Rp$, the unique maximal ideal generated by p. Let M be a comultiplication R-module such that $M = N \oplus P$ with submodules $N \neq (0)$ and $P \neq (0)$. Since R is a discrete valuation domain, by [11, Corollary 9.4], for any ideal I of R, there exists some positive integer n such that $I = m^n$.

Now, M is a comultiplication R-module. By [11, Corollary 9.4], there exists some positive integer m, n with $m < n$ such that

$$N = ann_M(m^n) \text{ and } P = ann_M(m^m).$$

This implies that

$$\begin{aligned} M &= N \oplus P \\ &= ann_M(m^n) + ann_M(m^m) \\ &= ann_M(m^n). \end{aligned}$$

Now, we have
$N \cap P = ann_M(m^n) \cap ann_M(m^m) = ann_M(m^n + m^m) \neq (0)$.

But this is a contradiction to the fact that $N \cap P = (0)$. Hence either $N = (0)$ or $P = (0)$.

Therefore, M is indecomposable.

Theorem 5.3. [2, Theorem 3.4] Let R be a pullback ring and let M be an indecomposable separated comultiplication R-module. Then M is isomorphic to one of the modules listed in Lemma 5.1.

Proof. Since M is separated comultiplication R-module, by [2, Proposition 3.3], $M = L \oplus N$, where L is one of the modules as described in (1) and N is one of the module described in (2) of Lemma 5.1. Now, M is indecomposable, either $M = L$ or $M = N$.

6. Conclusion

In this paper we have tried to present the whole results. These results are backbone of comultiplication modules. This paper is very useful for mathematical society.

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